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EXPONENTIAL LIFE TEST PROCEDURES WHEN THE DISTRIBUTION HAS MONOTONE FAILURE RATE

by

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This report is a revision of the February 1965 version, Boeing document D1-82-0415, bearing the same title. Many of the results of the earlier report have been extended using tools developed in "Inequalities for Linear Combinations of Order Statistics from Restricted Families," Boeing document D1-82-0506 and "Tolerance and Confidence Limits for Classes of Distributions Based on Failure Rate," Boeing document D1-82-0503.

ABSTRACT

A number of tests and estimates for mean life and other parameters derived under the exponential distribution assumption are studied under the alternative condition that the distribution has an increasing (decreasing) failure rate. The tests and estimates considered are, for the most part, based on censored and truncated samples. It is shown that the usual acceptance sampling procedures based on mean life generally favor the producer (consumer) in the IFR (DFR) case. However, acceptance sampling procedures based on the q^{th} quantile tend to favor the consumer in the IFR case when q is small. The usual estimates for the mean based on the exponential assumption are positively (negatively) biased when the distribution is IFR (DFR). Many of these results hold under the weaker assumption that the distribution has an increasing (decreasing) failure rate on the average.

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1. INTRODUCTION

In a fundamental paper in the literature of life testing, Epstein and Sobel (1953) introduce life test procedures based on the exponential distribution. (See also Epstein (1960a).) These procedures have been codified in a Department of Defense handbook H108 (1960) and are now widely employed. If the exponential procedures are based on a complete sample, the central limit law assures "robustness" for large sample sizes; i.e., the risk function is not sensitive to departures from the exponential distribution assumption. However, the robustness of exponential procedures based on censored or truncated samples has been questioned. Zelen and Dannemiller (1961) show by using Weibull distribution alternatives that these procedures may not be robust in testing for mean life. In particular, they show that the use of these procedures may result in substantially increasing the probability of accepting items having poor mean lives. Antelman and Savage in unpublished work point out that these exponential procedures used in testing hypotheses concerning the median seem to be robust. This is *not* the case in testing for low or high percentiles. Although the exponential statistics used in hypothesis testing may suffer in certain applications from lack of robustness, they do yield conservative tolerance and confidence limits for distributions with increasing failure rate in certain cases. This is suggested by the results of Zelen and Dannemiller and developed in detail in Barlow and Proschan (1966b) (hereafter referred to as BP (1966b).)

Since the usual tests and estimates based on the exponential assumption possess an undeniable computational simplicity and elegance, perhaps a further investigation of their properties when the true distribution is other than exponential is warranted. This paper interprets, in a life testing context, theoretical results concerning linear combinations of order statistics contained in Barlow and Proschan (1966a) (hereafter referred to as BP (1966a)). We consider statistical procedures based on an exponential distribution for a variety of life testing plans. In each case--censored sampling, censored sampling with replacement, or truncation--the usual estimate for the mean life is biased high (low) if the true distribution has an increasing (decreasing) failure rate. Properties of distributions with monotone failure rate are exploited to obtain bounds on the risk function when the usual decision rules based on an exponential assumption are used.

Mathematical Preliminaries. Let X denote a random variable with right continuous distribution F such that $F(0^-) = 0$. If F has density f, then $r(t) = f(t)/\bar{F}(t)$ is known as the failure rate, where $\bar{F} = 1-F$. Note that $r(t) = -\frac{d}{dt}\log[\bar{F}(t)]$ when a density exists. For this reason, we say that F is IFR (DFR) for increasing (decreasing) failure rate if $\log[\bar{F}(t)]$ is concave where finite [convex on $[0,\infty)$). Note that any IFR (DFR) distribution with specified mean can be expressed as the limit of continuous IFR (DFR) distributions with the same mean. Hence for many of our results it is sufficient to confine attention to continuous IFR (DFR) distributions and we do this in giving proofs.

A weaker restriction than monotonicity of the failure rate is also considered in some cases; namely, monotonicity of the failure rate average, $\frac{1}{t}\int_0^t r(u)du$. This motivates the definition of IFRA (DFRA) distributions. F is said to have an *increasing (decreasing) failure rate average*, i.e., to be IFRA (DFRA) if $-\log \bar{F}(x)/x$ is increasing (decreasing). Clearly, if F is IFR (DFR), then it is IFRA (DFRA). The virtues of the IFRA class of distributions are discussed in Birnbaum, Esary and Marshall (1966).

Properties of distributions with monotone failure rate are discussed in Barlow, Marshall and Proschan (1963) and in Barlow and Proschan (1965), Chapter 2. Tables of bounds for these distributions are given in Barlow and Marshall (1965). Additional bounds are given in Barlow and Marshall (1966). Distribution free life test sampling plans based on these bounds are discussed in Barlow and Gupta (1966).

Unless otherwise indicated, we denote ordered observations from a random sample of size n based on a random variable X by $X_{1,n} \leq \cdots \leq X_{n,n}$, and define $X_{o,n} = 0$. The second subscript is dropped when there is no danger of confusion. The term increasing (decreasing) is used for nondecreasing (nonincreasing). It will be convenient to let G denote the exponential distribution with mean, θ, i.e.

$$\bar{G}(x) = \begin{cases} 1 & x < 0 \\ e^{-\frac{x}{\theta}} & x \geq 0. \end{cases}$$

We shall consider distributions F that are IFR (DFR) or IFRA (DFRA) with an unknown scale parameter of interest, such as the mean or a specified quantile. We shall compare operating characteristic (OC) curves for the F distribution with corresponding OC curves for the exponential distribution as the scale parameter varies.

2. CENSORED SAMPLING TO ESTABLISH A SPECIFIED MEAN LIFE

Assume n items are placed on test simultaneously and testing is discontinued after the r^{th} $(r < n)$ failure. There is no replacement of failed items. If the distribution is exponential with density

$$g(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x \geq 0 \\ 0 & x < 0, \end{cases}$$

then

$$\hat{\theta}_{rn} = \frac{\sum_{1}^{r} X_i + (n-r)X_r}{r} \qquad (1 \leq r \leq n),$$

the total time on test divided by the number of failures observed, is the maximum likelihood and minimum variance unbiased estimate for θ based on the first r order statistics (Epstein and Sobel (1953)). The following decision rule based on $\hat{\theta}_{rn}$ is a uniformly most powerful test of the hypothesis $\theta = \theta_0$ against $\theta < \theta_0$ with Type I error equal to α.

$$\hat{\theta}_{rn} \begin{array}{c} \geq \\ \\ \leq \end{array} \left.\vphantom{\begin{array}{c} \geq \\ \\ \leq \end{array}}\right\} C(\theta_0) = \theta_0 \frac{\chi^2_\alpha(2r)}{2r} \begin{array}{l} \longrightarrow \text{accept} \\ \\ \longrightarrow \text{reject} \end{array} \qquad (2.1)$$

where $\chi_{\alpha}^{2}(2r)$ denotes the 100α per cent point of a chi-square distribution with $2r$ degrees of freedom and θ_0 is the mean life goal. Zelen and Dannemiller (1961) show that if the distribution is actually Weibull with increasing failure rate, then with $n = 28$, $r = 14$, $\theta_0 = 1000$ hours, and $\alpha = .10$, there is a high probability of accepting items with a mean life considerably less than $\theta_0 = 1000$ hours. This phenomenon is true in general when the distribution is actually IFR, as stated more precisely in

THEOREM 2.1: *If* F *is IFR (DFR) with mean* θ *and* G *is exponential with mean* θ*, then*

$$P_F\{\hat{\theta}_{rn} \geq C(\theta_0)|\theta\} \geq (\leq) \; P_G\{\hat{\theta}_{rn} \geq C(\theta_0)|\theta\} \tag{2.2}$$

for $\theta \geq \theta_0\chi_{\alpha}^{2}(2r)/2(n-r+1)$.

Inequality (2.2) follows from the result

$$P_F\{\sum_{1}^{r}(n-i+1)(X_i-X_{i-1}) \geq x\} \geq P_G\{\sum_{1}^{r}(n-i+1)(Y_i-Y_{i-1}) \geq x\}$$

for $x \leq (n-r+1)\theta$ and F IFR which is proved in BP (1966a), Corollary 5.5 and Theorem 4.6.

For small r, the coefficient of θ_0 is less than 1 (e.g., if $\alpha \leq e^{-1}$ and $r \leq (n+1)/2$, then $[\chi_{\alpha}^{2}(2r)/2(n-r+1)] \leq 1$); see BP (1966b). For the Zelen-Dannemiller example we can assert that the probability of

accepting items with true mean lives as low as $\frac{\chi^2_{.10}(28)}{30}$ 100% = 63% of the mean life goal is greater when the true distribution is IFR than when it is exponential. Hence we see that the *producer's* risk is controlled against IFR alternatives, but *not* the *consumer's* risk.

If we keep r fixed and increase n, inequality (2.2) is strengthened.

THEOREM 2.2: *If* F *is IFR (DFR) with mean* θ, *then*

$$P_F\{\hat{\theta}_{rn} \geq C(\theta_0)|\theta\} \leq (\geq) P_F\{\hat{\theta}_{r,n+1} \geq C(\theta_0)|\theta\} \tag{2.3}$$

for all $\theta > 0$.

Inequality (2.3) is proved in BP (1966a), Theorem 5.4. It follows from this result that $\hat{\theta}_{rn}$ as an estimate for θ is positively (negatively) biased when F is IFR (DFR).

COROLLARY 2.3: If F is IFR with mean θ, then

$$\theta \leq E_F[\hat{\theta}_{rn}] \leq E_F[\hat{\theta}_{r,n+1}] \leq \frac{(n+1)\theta}{r} . \tag{2.4}$$

If F is DFR with mean θ, then

$$0 \leq E_F[\hat{\theta}_{r,n+1}] \leq E_F\hat{\theta}_{rn} \leq \theta \tag{2.5}$$

for $1 \leq r < n$.

All inequalities are sharp.

Proof. The upper bound in (2.4) follows from the fact that for every sample realization

$$\sum_1^r X_i + (n-r)X_r \leq \sum_1^n X_i .$$

Since equality is attained with the distribution degenerate at θ (which is the limit of IFR distributions) the bound is sharp. The other inequalities in (2.4) follow from Theorem 2.2.

The lower bound in (2.5) is sharp as shown in Section 6 of BP (1966a). The other inequalities follow from Theorem 2.2.||

Actually it can be shown under the weaker condition of F IFRA (DFRA) that $E\hat{\theta}_{rn}$ is increasing (decreasing) in r for fixed n and hence $E\hat{\theta}_{rn} \geq (\leq)\ \theta$. (See Theorem 5.3, BP (1966a).)

3. CENSORED SAMPLING TO ESTABLISH A SPECIFIED QUANTILE LIFE

Suppose now that we wish to establish that the q^{th} quantile, ξ_q, exceeds some quantile life goal, say ξ_q°, with a Type I error $\leq \alpha$. We consider a censored sampling plan, as in Section 2. If the distribution is exponential, the decision rule is, using (2.1) and the relationship between ξ_q and θ,

$$\hat{\theta}_{rn} \left.\begin{matrix} \geq \\ \\ \leq \end{matrix}\right\} C'(\xi_q^\circ) = \frac{\xi_q^\circ \chi_\alpha^2(2r)}{-\,2r\ \ell n(1-q)} \quad \begin{matrix} \longrightarrow \text{accept} \\ \\ \longrightarrow \text{reject.} \end{matrix} \tag{3.1}$$

See Epstein (1960c). In reliability situations we will usually be interested in low percentiles; i.e. small values of q. For such values of q, the *consumer* rather than the *producer* will be protected under this decision rule, if the distribution is actually IFR. More precisely. we have

THEOREM 3.1: *If F is IFR (DFR) with q^{th} quantile ξ_q and G is exponential with q^{th} quantile ξ_q, then*

$$P_F\{\hat{\theta}_{rn} \geq C'(\xi_q^\circ)|\xi_q\} \leq (\geq)\ P_G\{\hat{\theta}_{rn} \geq C'(\xi_q^\circ)|\xi_q\} \tag{3.2}$$

when

$$\xi_q \leq \frac{\xi_q^\circ \chi_\alpha^2(2r)}{-2n\ \ell n(1-q)}\ .$$

Inequality (3.2) follows from Theorem 4.2 of BP (1966a). (See also Theorem 2.3 of BP (1966b).) Note that the coefficient of ξ_q° is greater than one when q is sufficiently small; i.e., $q \le 1 - \exp[-\chi_\alpha^2(2r)/2n]$, so that the consumer is protected against accepting material not meeting the quantile life goal.

If q is sufficiently large and decision rule (3.1) is used, then the *producer* rather than the *consumer* will be protected when the distribution is actually IFRA. More precisely,

THEOREM 3.2: *If F is IFRA (DFRA) with q^{th} quantile ξ_q and G is exponential with q^{th} quantile ξ_q, then*

$$P_F\{\hat{\theta}_{rn} \ge C'(\xi_q^\circ)\,|\,\xi_q\} \ge (\le)\; P_G\{\hat{\theta}_{rn} \ge C'(\xi_q^\circ)\,|\,\xi_q\} \tag{3.3}$$

when

$$\xi_q \ge \frac{\xi_q^\circ \chi_\alpha^2(2r)}{-2(n-r+1)\ln(1-q)} .$$

Inequality (3.3) follows from Theorem 3.4 of BP (1966a). (See also Theorem 3.3 of BP (1966b).) The coefficient of ξ_q° is less than one for large q; i.e., $q \ge 1 - \exp[-\chi_\alpha^2(2r)/2(n-r+1)]$. Hence for large values of q, the producer's risk is controlled but not the consumer's risk.

The analysis above points up the fact that one must be very careful in using exponential life tests based on censored samples. Depending on the objective chosen, either the producer or the consumer, but certainly not both simultaneously, can be protected if the distribution is actually IFR or DFR.

4. SAMPLING WITH REPLACEMENT

Another device to save time spent on experimentation is to replace failed items at the time of failure with new items until the total number of failed items reaches a fixed number, say r. Except for this restriction, the sampling plan is the same as in Sections 2 and 3. We shall see that the remarks concerning producer's and consumer's risks are even more applicable in this case.

Let X_i^* denote the time of the i^{th} failure when failed items are replaced. The maximum likelihood estimate for θ based on the exponential assumption is, in this case

$$\hat{\theta}_{rn}^{*} = \frac{1}{r}\,[nX_1^* + n(X_1^* + n(X_2^* - X_1^*) + \cdots + n(X_r^* - X_{r-1}^*)]$$

$$= nX_r^{*}{}' \; , \qquad (4.1)$$

the total time on test divided by the number of failures observed (Epstein, (1960b).

It will be convenient to introduce the following fictitious replacement policy:

Policy A: Replace a failed item with a good item of the same "age".

Let X_i^{**} denote the time of the i^{th} failure under this policy, and

$$\hat{\theta}_{rn}^{**} = \frac{1}{r} \sum_1^r n(X_i^{**} - X_{i-1}^{**}) = nX_r^{**}/r, \tag{4.2}$$

the total time on test divided by the number of failures observed.

LEMMA 4.1: If F is IFR (DFR), then

$$P_F\{\hat{\theta}_{rn} \geq x\} \leq (\geq) \; P_F\{\hat{\theta}_{rn}^{**} \geq x\}. \tag{4.3}$$

Proof. Assume F is IFR. The proof is by induction on r. Note that (4.3) is obviously true for $r = 1$ and all $n \geq 1$. Assume (4.3) is true for $r-1$ and all $n \geq r-1$. Let F_{1n} denote the distribution of X_{1n},

$$F_u(x) = [F(x+u) - F(u)]/\bar{F}(u)$$

(the distribution of an item of age u), and

$$T_{rn} = nX_{1n} + n(X_{2n}-X_{1n}) + \cdots + n(X_{rn}-X_{r-1,n}).$$

Then

$$P_F\{T_{rn} \geq x\} = \int_0^\infty P_F\{nX_{1n} + \cdots + (n-r+1)(X_{rn}-X_{r-1,n}) \geq x \mid nX_{1n} = u\}dF_{1n}(\tfrac{u}{n})$$

$$= \int_0^\infty P_{F_{\frac{u}{n}}}\{(n-1)X_{1,n-1} + \cdots + ((n-1)-(r-1)+1)(X_{r-1,n-1}-X_{r-2,n-1}) \geq x-u\}dF_{1n}(\tfrac{u}{n})$$

$$= \int_0^\infty P_{F_{\frac{u}{n}}}\{T_{r-1,n-1} \geq x-u\}dF_{1n}(\tfrac{u}{n})$$

$$\leq \int_0^\infty P_{F_{\frac{u}{n}}}\{T_{r-1,n} \geq x-u\}dF_{1n}(\tfrac{u}{n}).$$

The last inequality follows from Theorem 2.2.

Also

$$P\{T^{**}_{rn} \geq x\} = \int_0^\infty P_F\{n(X^{**}_{2n}-X^{**}_{1n}) + \cdots + n(X^{**}_{rn}-X^{**}_{r-1,n}) \geq x-u \mid nX^{*}_{1n} = u\}dF_{1n}(\tfrac{u}{n})$$

$$= \int_0^\infty P_{F_{\frac{u}{n}}}\{T^{**}_{r-1,n} \geq x-u\}dF_{1n}(\tfrac{u}{n}).$$

By the induction assumption

$$P_{F_{\frac{u}{n}}}\{T_{r-1,n} \geq x-u\} \leq P_{F_{\frac{u}{n}}}\{T^{**}_{r-1,n} \geq x-u\}.$$

Hence

$$\int_0^\infty P_{F_{\frac{u}{n}}}\{T_{r-1,n-1} \geq x-u\}dF_{1n}(\tfrac{u}{n}) \leq \int_0^\infty P_{F_{\frac{u}{n}}}\{T^{**}_{r-1,n} \geq x-u\}dF_{1n}(\tfrac{u}{n})$$

which proves (4.3).

An analogous proof holds in the DFR case.||

We may now show that the usual estimate $\hat{\theta}^*_{rn}$ under censored sampling with replacement is stochastically larger than the corresponding estimate $\hat{\theta}_{rn}$ under censored sampling without replacement.

THEOREM 4.2: *If* F *is IFR (DFR), then*

$$P_F\{\hat{\theta}_{rn} \geq x\} \leq (\geq)\ P_F\{\hat{\theta}^*_{rn} \geq x\}.$$

Proof. Assume F is IFR. By Lemma 4.1 it is sufficient to show that

$$P_F\{\hat{\theta}^{**}_{rn} \geq x\} \leq P_F\{\hat{\theta}^*_{rn} \geq x\}. \tag{4.4}$$

To show this, consider the set of n renewal processes generated by successive failures and their replacements. Let $N^*_i(t)$ $(N^{**}_i(t))$ denote the number of replacements in the i^{th} renewal process under the original replacement policy (Policy A). Thus to show (4.4) we need only show

$$P_F\{T^{**}_{rn} \geq x\} \leq P_F\{T^*_{rn} \geq x\}$$

or, equivalently

$$P_F\{N^{**}_1(x) + \cdots + N^{**}_n(x) \leq r\} \leq P_F\{N^*_1(x) + \cdots + N^*_n(x) \leq r\}.$$

Since life lengths in different renewal processes are independent, we need only show

$$P_F\{N_i^{**}(x) \le r\} \le P_F\{N_i^{*}(x) \le r\}. \tag{4.5}$$

Let $\{U_i\}_{i=1}^{\infty}$ denote a renewal process with underlying distribution F, IFR. Let $\{V_i\}_{i=1}^{\infty}$ denote a sequence of random variables where V_1 has distribution F, and V_i given V_{i-1} has distribution

$$F_{V_i|V_{i-1}}(t) = \frac{F(t+V_{i-1}) - F(V_{i-1})}{\bar{F}(V_{i-1})} .$$

Then (4.5) is equivalent to

$$P_F\{N_i^{**}(x) \ge r\} = P_F\{V_1 +\cdots+ V_r \le x\} \ge P_F\{U_1 +\cdots+ U_r \le x\} = P_F\{N_i^{*}(x) \ge r\} \tag{4.6}$$

To prove (4.6) we use induc ı on r. Clearly (4.6) is true for $r = 1$. Suppose it is true for $r-1$. Then

$$P\{V_1 +\cdots+ V_r \le x\} = \int_0^x P_{F_u}\{V_1 +\cdots+ V_{r-1} \le x-u\}dF(u)$$

$$\ge \int_0^x P_{F_u}\{U_1 +\cdots+ U_{r-1} \le x-u\}dF(u) \ge \int_0^x P_F\{U_1 +\cdots+ U_{r-1} \le x-u\}dF(u).$$

The first inequality follows from the induction assumption and the second from the IFR property that

$$F_u(t) \geq F(t)$$

for all $u > 0$ and all $t \geq 0$.

An analogous proof holds in the DFR case.||

An obvious consequence of Theorem 4.2 is that $E\hat{\theta}^*_{rn} \geq (\leq)\ E\hat{\theta}_{rn} \geq (\leq)\ \theta$ in the IFR (DFR) case.

5. TRUNCATED LIFE TEST PLANS

A common practice in life testing is to truncate the experiment at a pre-assigned time, say T, and note the number of failures. If the number of failures is less than or equal to an acceptance number c, the life length goal is considered to have been demonstrated; otherwise, not. Such sampling plans have been discussed by Sobel and Tischendorf (1959) for the exponential case. In Barlow and Gupta (1966) such sampling plans are considered for IFR (DFR) distributions using known bounds for these distributions. It is pointed out that in the IFR case we have to test beyond the mean (or quantile) life goal to protect the consumer, whereas we can protect the producer by testing for a time less than the mean (or quantile) life goal. Sampling experiments by Zelen and Dannemiller (1961) show that the exponential procedures will perform poorly when the underlying distribution is Weibull if the truncation time, T, is less than the mean life goal.

Bartholomew (1963) considers truncated life test plans for the exponential distribution where the times of failures occurring before T, the truncation time, are assumed known. Assuming at least one failure has occurred by time T, the maximum likelihood estimate for the mean, θ, is

$$\hat{\theta}(T) = \frac{1}{r} \sum_{1}^{n} [T - a_i(T-Y_i)]$$

where $Y_1 \leq Y_2 \leq \cdots \leq Y_n$ denotes an ordered sample from an exponential distribution and

$$a_i = \begin{cases} 1 & \text{if } Y_i \leq T \\ 0 & \text{otherwise,} \end{cases}$$

and $r = \sum_{1}^{n} a_i$. The distribution of this statistic is computed in Bartholomew (1963). As in the censorship case, an acceptance test based on this statistic will favor the producer if the true distribution is IFR and T is not too large.

THEOREM 5.1: *If F is IFR (DFR) with mean θ, $T < \theta$, and $G(t) = 1 - e^{-t/\theta}$, then*

$$P_F\{\hat{\theta}(T) \geq x | r \geq 1\} \geq (\leq) P_G\{\hat{\theta}(T) \geq x | r \geq 1\}$$

for all $x \geq 0$.

Proof. Assume F is IFR. Let $Y_i = -\theta \log \bar{F}(X_i)$ and let

$$b_i = \begin{cases} 1 & \text{if } Y_i \le T \\ 0 & \text{otherwise.} \end{cases}$$

Since $G(x)$ crosses $F(x)$ to the right of θ and from above (Barlow and Proschan (1965), Chapter 2), we see that $X_i < T$ implies $Y_i < T$, and hence $a_i \le b_i$. Thus if $a_i = 1$, then $b_i = 1$ and so $T-X_i \le T-Y_i$. Hence if $\sum_1^n a_i \ge 1$,

$$\frac{\sum_1^n [T-a_i(T-X_i)]}{\sum_1^n a_i} \ge \frac{\sum_1^n [T-b_i(T-Y_i)]}{\sum_1^n b_i} .$$

By Lemma 1, page 73, Lehman (1959), it follows that

$$P_F\{\hat{\theta}(T) \ge x \mid r \ge 1\} \ge P_G\{\hat{\theta}(T) \ge x \mid r \ge 1\}$$

for all $x \ge 0$ and $T < \theta$.

A similar proof holds in the DFR case.||

Bartholomew (1957) uses the following estimate for θ in the exponential case:

$$\hat{\theta}(T) = \begin{cases} \frac{1}{r}\sum_{1}^{r} Y_i + (n-r)T & \text{if} \quad r > 0 \\ nT & \text{if} \quad r = 0. \end{cases}$$

Note that for $r > 0$, this is the maximum likelihood estimate discussed above. Then

$$E[\hat{\theta}(T)] = \theta - \frac{\text{cov}(r,\hat{\theta}(T))}{1-\exp(-\frac{T}{\theta})} > \theta$$

since r and $\hat{\theta}(T)$ are negatively correlated, Bartholomew (1957). Thus the estimate is positively biased when the underlying distribution is exponential. In BP (1966a) it is shown that

$$E_F\left[\sum_{1}^{r} X_i + (n-r)T\right] \geq (\leq)\ E_G\left[\sum_{1}^{r} Y_i + (n-r)T\right]$$

when F is IFRA (DFRA); T is not required to be $< \theta$ as in Theorem 5.1. This would suggest that the estimate is even further biased when the underlying distribution is IFRA.

<u>Inverse Binomial Sampling</u>. Nadler (1960) has considered the following type of sampling: An item having life distribution F with mean θ is put on test until it fails or time T has elapsed; at this time the item is replaced by a fresh item. This is repeated sequentially until r actual failures are observed. The number, N, of items that

have to be tested until r actual failures are obtained is a random variable. Let $X_1, X_2, \ldots, X_n, \ldots$ denote a sequence of independent random variables with distribution F. Let

$$Z_i = \begin{cases} X_i & \text{if } X_i \leq T \\ T & \text{otherwise,} \end{cases}$$

so that Z_i is the time on test of the i^{th} item. Nadler showed that when $F(x) = 1-e^{-x/\theta}$, a sufficient, unbiased estimate of θ is

$$\hat{\theta}_r(T) = \frac{1}{r} \sum_1^N Z_i .$$

We compare $\hat{\theta}_r(T)$ with related statistics in

THEOREM 5.1: *If F is IFR (DFR) with mean θ, then*

$$P_F\{\hat{\theta}_r(T) \geq x\} \geq (\leq)\ P_F\{\hat{\theta}_{rr} \geq x\} \tag{5.1}$$

for all $x \geq 0$, and hence

$$E_F[\hat{\theta}_r(T)] \geq (\leq)\ \theta .$$

If, in addition, $T < \theta$, then

$$P_F\{\hat{\theta}_r(T) \geq x\} \geq (\leq)\ P_G\{\hat{\theta}_r(T) \geq x\} = P_G\{\hat{\theta}_{rr} \geq x\} \tag{5.2}$$

for all $x \geq 0$.

Proof. Assume F is IFR. Let W_i denote the time between the (i-1)st and i^{th} actual failure following an inverse binomial sampling policy. Note that $\frac{1}{r}\sum_1^r W_i = \hat{\theta}_r(T)$. Then for $nT \leq x < (n+1)T$,

$$P[W_i \geq x] = [\bar{F}(T)]^n[\bar{F}(x-nT)] \geq \bar{F}(x) = P[X_i \geq x];$$

the inequality follows from the fact that F is IFR. Hence

$$P_F\left[\sum_1^r W_i \geq x\right] \geq P_F\left[\sum_1^r X_i \geq x\right],$$

proving (5.1).

To show (5.2), let $Y_i = -\log \bar{F}(X_i)$. Note that $Y_i \leq X_i$ if $X_i \leq T < \theta$. It follows that if N' denotes the number of Y_i's which must be examined to obtain r values each less than T, we see that $N' \leq N$. Therefore, for each sample,

$$\sum_1^{N'} Z_i' \leq \sum_1^{N} Z_i,$$

where

$$Z_i' = \begin{cases} Y_i & \text{if } Y_i \leq T \\ T & \text{otherwise.} \end{cases}$$

(5.2) follows from Lemma 1, page 73 of Lehman (1959).||

6. BOUNDS ON ESTIMATES FOR THE RELIABILITY FUNCTION

In this section we obtain a lower bound for the expected value of two estimates of reliability appropriate under the exponential assumption, when the distribution is actually IFR. Assume that a censored sampling plan is followed, as in Section 2.

The minimum variance unbiased estimate for $R(t) = \bar{F}(t)$ (t is fixed) under the exponential assumption is

$$\hat{R}_1(t) = \begin{cases} \left(1 - \frac{t}{Z_r}\right)^{r-1} & \text{if } t < Z_r \\ 0 & \text{otherwise,} \end{cases}$$

where $Z_r = \sum_1^r X_i + (n-r)X_r$, the total time on test. For a discussion of such minimum variance unbiased estimates, see Tate (1959). Then, under the exponential assumption with the mean θ taken to be 1 for convenience, Z_r has density

$$g_r(y) = \frac{y^{r-1}e^{-y}}{(r-1)!} .$$

We obtain a lower bound for $\hat{R}_1(t)$ in

THEOREM 6.1: *If* F *is IFR with mean* $\theta = 1$ *and* $t < \theta$, *then*

$$E[\hat{R}_1(t) \geq \int_t^1 \left[1 - \frac{t}{y}\right]^{r-1} g_r(y)dy + (1-t)^{r-1} \int_1^\infty g_r(y)dy.$$

Proof. Let $\phi^{-1}(x) = -\log \bar{F}(x)$. Then ϕ is concave, increasing and $\phi(0) = 0$. If Y_i is the i^{th} order statistic in a sample of n from an exponential distribution with mean 1, then $X_i = \phi(Y_i)$ is distributed as the i^{th} order statistic from F. Furthermore,

$$Z_r = \sum_1^r \phi(Y_i) + (n-r)\phi(Y_r)$$

$$\geq \phi\left[\sum_1^r Y_i + (n-r)Y_r\right] = \phi(Z_r^*).$$

Therefore,

$$\hat{R}_1(t) = \begin{cases} \left(1 - \dfrac{t}{Z_r}\right)^{r-1} & \text{if } t < Z_r \\ 0 & \text{otherwise} \end{cases}$$

$$\geq \hat{R}_1^*(t) = \begin{cases} \left[1 - t/\phi(Z_r^*)\right]^{r-1} & \text{if } t < \phi(Z_r^*) \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\phi(y) \geq \begin{cases} y & y < 1 \\ 1 & y \geq 1, \end{cases}$$

we have

$$E[\hat{R}_1(t)] \geq \int_t^1 \left[1 - \frac{t}{y}\right]^{r-1} g_r(y)dy + \int_1^\infty (1-t)^{r-1} g_r(y)dy. ||$$

The maximum likelihood estimator for $R(t)$ under the exponential assumption is

$$\hat{R}_2(t) = \exp[-t/\hat{\theta}_{rn}]$$

where $\hat{\theta}_{rn}$ is defined in Section 2. Pugh (1963) has shown that under the exponential assumption $\hat{R}_2(t)$ is negatively biased when the true reliability $R(t) > \frac{1}{e} \sim .368$. Assuming F is IFR we can obtain a lower bound on $E[\hat{R}_2(t)]$.

THEOREM 6.2: *If* F *is IFR with mean* $\theta = 1$, *then*

$$E[\hat{R}_2(t)] \geq \int_0^1 e^{-\frac{tr}{y}} g_r(y)dy + e^{-tr} \int_1^\infty g_r(y)dy.$$

The proof parallels that of Theorem 6.1.

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